Exemplo:

$$U(x_1, x_2) = x_1^a x_2^b, \quad a, b > 0$$

$$\text{Max } U(x_1, x_2)$$
$$\text{suj. } p_1 x_1 + p_2 x_2 = m$$

$$UMg_1 = a x_1^{a-1} x_2^b$$
$$UMg_2 = b x_1^a x_2^{b-1}$$
$$\frac{UMg_1}{UMg_2} = \frac{a}{b}\frac{x_2}{x_1}$$

Funções de demanda:

$$x_1^*(p_1, p_2, m) = \frac{a}{a+b}\frac{m}{p_1} \quad ; \quad x_2^*(p_1, p_2, m) = \frac{b}{a+b}\frac{m}{p_2}$$

Função de utilidade indireta:

$$V(p_1, p_2, m) = \left(\frac{a}{p_1}\right)^a \left(\frac{b}{p_2}\right)^b \left(\frac{m}{a+b}\right)^{a+b}$$

Funções de demanda compensada e de dispêndio:

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2$$
$$\text{suj. a } U(x_1, x_2) \geq \underline{u}$$

$$\begin{cases} \frac{a}{b}\frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{b}{a}\frac{p_1}{p_2} x_1 \\ x_1^a x_2^b = u \Rightarrow x_1^a\left(\frac{b}{a}\frac{p_1}{p_2}x_1\right)^b = u \Rightarrow h_1(p_1, p_2, u) = \left(\frac{a}{b}\frac{p_2}{p_1}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \end{cases}$$

$$h_2(p_1, p_2, u) = \left(\frac{b}{a}\frac{p_1}{p_2}\right)^{\frac{a}{a+b}} u^{\frac{1}{a+b}}$$

$$e(p_1, p_2, u) = p_1 h_1(p_1, p_2, u) + p_2 h_2(p_1, p_2, u)$$
$$= p_1\left(\frac{a}{b}\frac{p_2}{p_1}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} + p_2\left(\frac{b}{a}\frac{p_1}{p_2}\right)^{\frac{a}{a+b}} u^{\frac{1}{a+b}}$$

$$e(p_1,p_2,u) = p_1\left(\frac{a}{b}\frac{p_2}{p_1}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} + p_2\left(\frac{b}{a}\frac{p_1}{p_2}\right)^{\frac{a}{a+b}} u^{\frac{1}{a+b}}$$

$$= \left(\frac{a}{b}\right)^{\frac{b}{a+b}} p_2^{\frac{b}{a+b}} p_1^{\frac{a}{a+b}} u^{\frac{1}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}} p_2^{\frac{b}{a+b}} p_1^{\frac{a}{a+b}} u^{\frac{1}{a+b}}$$

$$= u^{\frac{1}{a+b}} p_1^{\frac{a}{a+b}} p_2^{\frac{b}{a+b}} \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] \checkmark$$

$$= u^{\frac{1}{a+b}} p_1^{\frac{a}{a+b}} p_2^{\frac{b}{a+b}} \left[\frac{a+b}{b^{\frac{b}{a+b}}\, a^{\frac{a}{a+b}}}\right]$$

$$\boxed{e(p_1,p_2,u) = (a+b)\, u^{\frac{1}{a+b}} \left(\frac{p_1}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}}}$$

1) Da função de utilidade indireta à função de dispêndio:

$$\boxed{V(p_1, p_2, m) = \left(\frac{a}{p_1}\right)^a \left(\frac{b}{p_2}\right)^b \left(\frac{m}{a+b}\right)^{a+b}}$$

$$\boxed{V[p_1, p_2, e(p_1, p_2, u)] = u}$$

$$\left(\frac{a}{p_1}\right)^a \left(\frac{b}{p_2}\right)^b \left(\frac{e(p_1, p_2, u)}{a+b}\right)^{a+b} = u$$

$$=$$

$$\boxed{e(p_1, p_2, u) = (a+b)\, u^{\frac{1}{a+b}} \left(\frac{p_1}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}}}$$

Da função de dispêndio à função de utilidade indireta:

$$e(p_1, p_2, u) = (a+b)\, u^{\frac{1}{a+b}} \left(\frac{p_1}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}} \quad \leftarrow$$

$$e\left[p_1, p_2, V(p_1, p_2, m)\right] = m \quad \Leftarrow$$

$$(a+b)\left[V(p_1, p_2, m)\right]^{\frac{1}{a+b}} \left(\frac{p_1}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}} = m$$

$$\boxed{V(p_1, p_2, m) = \left(\frac{a}{p_1}\right)^{a} \left(\frac{b}{p_2}\right)^{b} \left(\frac{m}{a+b}\right)^{a+b}}$$

Das funções de demanda marshalliana e de dispêndio à função de demanda compensada:

$$x_1^*(p_1, p_2, m) = \frac{a}{a+b}\,\frac{m}{p_1} \;;\; e(p_1, p_2, u) = (a+b)\,u^{\frac{1}{a+b}}\left(\frac{p_1}{a}\right)^{\frac{a}{a+b}}\left(\frac{p_2}{b}\right)^{\frac{b}{a+b}}$$

$$x_i^*[p_1, p_2, e(p_1, p_2, u)] = h_1(p_1, p_2, u)$$

$$h_1(p_1, p_2, u) = \frac{a}{\cancel{a+b}}\;\frac{\cancel{(a+b)}\,u^{\frac{1}{a+b}}\left(\frac{p_1}{a}\right)^{\frac{a}{a+b}}\left(\frac{p_2}{b}\right)^{\frac{b}{a+b}}}{p_1}$$

$$= u^{\frac{1}{a+b}}\left(\frac{a}{p_1}\,\frac{p_2}{b}\right)^{\frac{b}{a+b}}$$

① Das funções de demanda compensada e da função de utilidade indireta à função de demanda marshalliana:

$$h_2(p_1, p_2, u) = u^{\frac{1}{a+b}} \left(\frac{b}{a}\frac{p_1}{p_2}\right)^{\frac{a}{a+b}}$$

$$V(p_1, p_2, m) = \left(\frac{a}{p_1}\right)^{a}\left(\frac{b}{p_2}\right)^{b}\left(\frac{m}{a+b}\right)^{a+b}$$

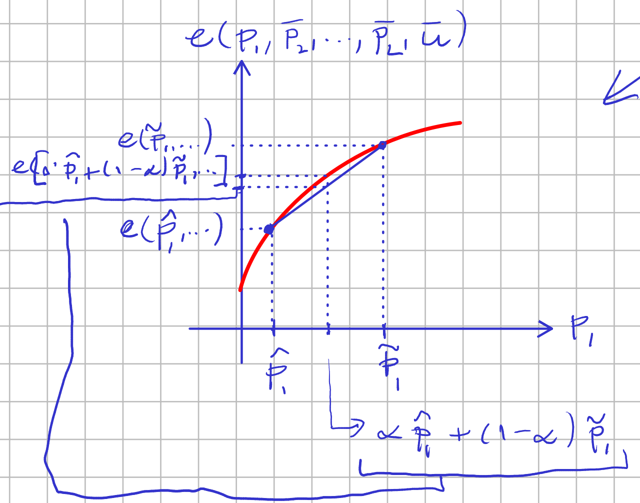
$$h_2\left[p_1, p_2, V(p_1, p_2, m)\right] = x_2^*(p_1, p_2, m)$$

$$x_2^*(p_1, p_2, m) = \left(\frac{\cancel{a}}{\cancel{p_1}}\right)^{\frac{a}{a+b}}\left(\frac{b}{p_2}\right)^{\frac{b}{a+b}}\left(\frac{m}{a+b}\right)^{\cancel{\frac{a+b}{a+b}}\,1}\left(\frac{b}{\cancel{a}}\frac{p_1}{p_2}\right)^{\frac{a}{a+b}} = \frac{b}{a+b}\frac{m}{p_2}$$

- A função de dispêndio é côncava em relação aos preços, ou seja, para quaisquer vetores de preços positivos, $\hat{p}$ e $\tilde{p}$, qualquer nível de utilidade u e qualquer $0 < \alpha < 1$,

$$\underbrace{\alpha e(\hat{p}, u) + (1-\alpha) e(\tilde{p}, u)} \leq \underbrace{e[\alpha \hat{p} + (1-\alpha)\tilde{p}, u]}$$

$$\alpha e(\hat{p}_1, \dots) + (1-\alpha) e(\tilde{p}_1, \dots)$$



$$e(\hat{p}, u) \leq \hat{p} \cdot h[\alpha \hat{p} + (1-\alpha)\tilde{p}, u] \qquad (\alpha)$$

$$e(\tilde{p}, u) \leq \tilde{p} \cdot h[\alpha \hat{p} + (1-\alpha)\tilde{p}, u] \qquad (1-\alpha)$$

$$\alpha e(\hat{p}, u) + (1-\alpha) e(\tilde{p}, u) \leq \alpha \hat{p} \cdot h[\alpha \hat{p} + (1-\alpha)\tilde{p}, u] + (1-\alpha)\tilde{p} \cdot h[\alpha \hat{p} + (1-\alpha)\tilde{p}, u]$$

$$\alpha e(\hat{p}, u) + (1-\alpha) e(\tilde{p}, u) \leq \underbrace{[\alpha \hat{p} + (1-\alpha)\tilde{p}] \cdot h[\alpha \hat{p} + (1-\alpha)\tilde{p}, u]}_{= e[\alpha \hat{p} + (1-\alpha)\tilde{p}, u]}$$

$$\alpha e(\hat{p}, u) + (1-\alpha) e(\tilde{p}, u) \leq e[\alpha \hat{p} + (1-\alpha)\tilde{p}, u]$$

Da função de dispêndio à função de demanda compensada:

$$e(p_1, p_2, u) = \underbrace{(a+b)\, u^{\frac{1}{a+b}} \left(\frac{p_1}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}}}$$

$$h(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{a^{\frac{b}{a+b}}}{\cancel{a+b}}\, \cancel{(a+b)}\, \frac{u^{\frac{1}{a+b}}}{\cancel{a}^{\frac{a}{a+b}}}\, p_1^{\frac{-b}{a+b}} \left(\frac{p_2}{b}\right)^{\frac{b}{a+b}} = \underbrace{u^{\frac{1}{a+b}} \left(\frac{a}{b}\,\frac{p_2}{p_1}\right)^{\frac{b}{a+b}}}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
x^*(p,m) \arrow[r, shift left, "{V(p,m) = U[x^*(p,m)]}"] \arrow[d, shift left, "{h(p,u) = x[p, e(p,u)]}"] & V(p,m) \arrow[l, shift left, "{x_i^*(p,m) = -\frac{\frac{\partial V(p,m)}{\partial p_i}}{\frac{\partial V(p,m)}{\partial m}}}"] \arrow[d, shift left, "{V[p, e(p,u)] = u}"] \\
h(p,u) \arrow[u, shift left, "{x^*(p,m) = h[p, V(p,m)]}"] \arrow[r, shift left, "{e(p,u) = p \cdot h(p,u)}"] & e(p,u) \arrow[l, shift left, "{h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}}"] \arrow[u, shift left, "{e[p, V(p,m)] = m}"]
\end{tikzcd}
$$